Linear Algebra I

09/10/2023

1 Systems of linear equations $(1+6+3+(1+6+3)=20$ pts$)$

Let a be a scalar. Consider the following system of linear equations in the unknowns x, y, and z:

$$\begin{aligned} x+2y-3z&=4\\ 3x-y+5z&=-2\\ 4x+y+(a^2-14)z&=a+6. \end{aligned}$$

(a) Write down the corresponding augmented matrix .

(b) By performing elementary row operations, put the augmented matrix into row echelon form.

(c) Determine all values of a so that the system is consistent.

(d) For $a=5$,

  (i) determine the *lead* and *free* variables.

  (ii) put the augmented matrix into *reduced* row echelon form by performing elementary row operations.

  (iii) find the solution set.

Required Knowledge: **Gauss-Jordan elimination, row operations, reduced row echelon form, notions of lead/free variables.**

Solution:

1a: The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 4\\ 3 & -1 & 5 & -2\\ 4 & 1 & a^2-14 & a+6 \end{bmatrix}.$$

1b: To put the augmented matrix into row echelon form, we apply elementary row operations:

$$\begin{bmatrix} 1 & 2 & -3 & 4\\ 3 & -1 & 5 & -2\\ 4 & 1 & a^2-14 & a+6 \end{bmatrix} \xrightarrow{\substack{\textcircled{2}\leftarrow\textcircled{2}-3\cdot\textcircled{1}\\ \textcircled{3}\leftarrow\textcircled{3}-4\cdot\textcircled{1}}} \begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -14\\ 0 & -7 & a^2-2 & a-10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -14\\ 0 & -7 & a^2-2 & a-10 \end{bmatrix} \xrightarrow{\textcircled{3}\leftarrow\textcircled{3}-1\cdot\textcircled{2}} \begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -14\\ 0 & 0 & a^2-16 & a+4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -14\\ 0 & 0 & a^2-16 & a+4 \end{bmatrix} \xrightarrow{\textcircled{2}\leftarrow-\frac{1}{7}\cdot\textcircled{2}} \begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & 1 & -2 & 2\\ 0 & 0 & a^2-16 & a+4 \end{bmatrix}$$

Now, we distinguish two cases: $a^2-16\neq 0$ or $a^2-16=0$.

If $a^2-16\neq 0$, then $a\neq 4$ and one more elementary row operation puts the matrix into row echelon form:

$$\begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -14\\ 0 & 0 & a^2-16 & a+4 \end{bmatrix} \xrightarrow{\textcircled{3}\leftarrow\frac{1}{a^2-16}\cdot\textcircled{3}} \begin{bmatrix} 1 & 2 & -3 & 4\\ 0 & 1 & -2 & 2\\ 0 & 0 & 1 & \frac{1}{a-4} \end{bmatrix}.$$

If $a^2 - 16 = 0$, then either $a = -4$ or $a = 4$. For $a = -4$, the matrix is already in the row echelon form:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $a = 4$, one more elementary row operation yields the row echelon form:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -14 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{\textcircled{3} \leftarrow \frac{1}{8} \cdot \textcircled{3}} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1c: From (1b), we see that the system is consistent if and only if $a \neq 4$.

1d(i): For $a = 5$, we obtained in (1b) the following row echelon matrix.

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then, there no free variables and hence all variables are free.

1d(ii): To obtain the reduced row echelon form, we apply elementary row operations:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{\textcircled{2} \leftarrow \textcircled{2} + 2 \cdot \textcircled{3} \\ \textcircled{1} \leftarrow \textcircled{1} + 3 \cdot \textcircled{3}}} \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} - 2 \cdot \textcircled{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1d(iii): The unique solution is $x = -1$, $y = 4$, and $z = 1$.

2 Matrix multiplication (5 + 10 = 15 pts)

Let $A \in \mathbb{F}^{p\times q}$ and $B \in \mathbb{F}^{q\times p}$. Prove or disprove the statements:

(a) $\det(AB) = \det(BA)$

(b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (For $M \in \mathbb{F}^{m\times m}$, $\mathrm{tr}(M) := \sum_{k=1}^{m}[M]_{kk}$.)

Required Knowledge: **Matrix multiplication, determinant.**

Solution:

2a: We know that $\det(AB) = \det(BA)$ if $p = q$. When $p \neq q$, however, this relation does not hold in general. A counter example can be obtained by taking

$$A = \begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 1\end{bmatrix}.$$

Indeed, these choices yield

$$AB = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \quad \text{and} \quad BA = 2.$$

Note that $\det(AB) = 0 \neq 2 = \det(BA)$.

2b: From the definition of the matrix multiplication, we know that

$$[AB]_{ij} = \sum_{k=1}^{q}[A]_{ik}[B]_{kj} \quad \text{and} \quad [BA]_{ij} = \sum_{\ell=1}^{p}[B]_{i\ell}[A]_{\ell j}.$$

Therefore, we have

$$\mathrm{tr}(AB) = \sum_{i=1}^{p}[AB]_{ii} = \sum_{i=1}^{p}\sum_{k=1}^{q}[A]_{ik}[B]_{ki} = \sum_{k=1}^{q}\sum_{i=1}^{p}[B]_{ki}[A]_{ik} = \sum_{k=1}^{q}[BA]_{kk} = \mathrm{tr}(BA).$$

3 Determinants (10 + 15 = 25 pts)

Let $M(n) \in \mathbb{R}^{n\times n}$ be given by

$$[M(n)]_{ij} = \begin{cases} 1 & \text{if } |i-j| \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

For instance,

$$M(5) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Compute the determinant of $M(n)$ for $n \in \{1, 2, 3\}$.

(b) Find real numbers a, b such that $\det\big(M(n+2)\big) = a\det\big(M(n+1)\big) + b\det\big(M(n)\big)$ for all $n \geqslant 1$.

Required Knowledge: **Matrix multiplication, nonsingular matrices, and inverse matrix.**

Solution:

3a: We have

$$M(1) = 1, \quad M(2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad M(3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, we see that $\det(M(1)) = 1$ since $M(1)$ is a scalar, $\det(M(2)) = 0$ since $M(2)$ has two identical rows, and $\det(M(1)) = 1 - 1 - 1 = -1$ due to Sarrus' rule.

3b: Let $M = M(n+2)$. The cofactor expansion of the determinant M along the first row yields:

$$\det(M(n+2)) = \det(M_{11}) - \det(M_{21}).$$

Note that M_{11} is nothing but $M(n+1)$ and the first column of M_{12} is the first column of I_{n+1}. By the cofactor expansion of M_{12} along its first column, we see that $\det(M_{12}) = \det((M_{12})_{11})$. Note that matrix $(M_{12})_{11}$ is the matrix obtained from M by deleting the first two rows and columns. Therefore, we see that $\det(M_{12}) = \det(M(n))$. Hence, we obtain

$$\det(M(n+2)) = \det(M(n+1)) - \det(M(n))$$

showing that $a = 1$ and $b = -1$.

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4 Nonsingular matrices (15 + 15 = 30 pts)

Let a, b, c, d be scalars and $M \in \mathbb{F}^{m\times m}$. Consider the matrix

$$N = \begin{bmatrix} aM & bM \\ cM & dM \end{bmatrix}.$$

(a) Show that N is nonsingular if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and M are nonsingular.

(b) Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and M are nonsingular. Find the inverse of N.

REQUIRED KNOWLEDGE: **Nonsingularity and partitioned matrices.**

SOLUTION:

4a: To prove the 'only if' part, let e, f be such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \mathbf{0}_2$. Then, we have

$$ae + bf = 0 \quad \text{and} \quad ce + df = 0.$$

Now, let $\boldsymbol{x}$ be a nonzero vector in $\mathbb{F}^m$. Note that

$$N\begin{bmatrix} e\boldsymbol{x} \\ f\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} (ae+bf)\boldsymbol{x} \\ (ce+df)\boldsymbol{x} \end{bmatrix} = \mathbf{0}_{2m}$$

Since N is nonsingular, we can conclude that $e\boldsymbol{x} = f\boldsymbol{x} = \mathbf{0}_m$. As $\boldsymbol{x}$ is a nonzero vector, we see that $e = f = 0$. Consequently, the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular. What remains to be proven is the nonsingularity of M. Let $\boldsymbol{x} \in \mathbb{F}^m$ be such that $M\boldsymbol{x} = \mathbf{0}_m$. Note that

$$N\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x} \end{bmatrix} = \mathbf{0}_{2m}.$$

From the nonsingularity of N, it follows that $\boldsymbol{x} = \mathbf{0}_m$ and hence M is nonsingular.

To prove the 'if' part, let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^m$ be such that

$$N\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \mathbf{0}_{2m}.$$

Since

$$N\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} aM\boldsymbol{x} + bM\boldsymbol{y} \\ cM\boldsymbol{x} + dM\boldsymbol{y} \end{bmatrix},$$

we see that

$$aM\boldsymbol{x} + bM\boldsymbol{y} = \mathbf{0}_m \tag{1}$$
$$cM\boldsymbol{x} + dM\boldsymbol{y} = \mathbf{0}_m. \tag{2}$$

If we multiply both sides of (1) by d, those of (2) by $-b$ and add them up, we obtain

$$(ad - bc)M\boldsymbol{x} = \mathbf{0}_m. \tag{3}$$

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular, its determinant $ad - bc \neq 0$, it then follows from (3) that $M\boldsymbol{x} = \mathbf{0}_m$. As M is nonsingular, we can conclude that $\boldsymbol{x} = \mathbf{0}_m$. By proceeding in a similar fashion, one can multiply (1) by $-c$ and (2) by a to obtain $(ad - bc)M\boldsymbol{y} = \mathbf{0}_m$. This would lead to $\boldsymbol{y} = 0_m$ and proves that N is nonsingular.

4b: Let $U, V, W, Z \in \mathbb{F}^{m\times m}$ be such that

$$N^{-1} = \begin{bmatrix} U & V \\ W & Z \end{bmatrix}.$$

By using $NN^{-1} = I_{2m}$, we see that

$$aMU + bMW = I_m \tag{4}$$
$$aMV + bMZ = 0_{m,m} \tag{5}$$
$$cMU + dMW = 0_{m,m} \tag{6}$$
$$cMV + dMZ = I_m \tag{7}$$

From (4) and (6), we see that $d(aMU + bMW) - b(cMU + dMW) = dI_m$, or equivalently $(ad - bc)MU = dI_m$. Since $ad - bc \neq 0$ due to the nonsingularity of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and M is nonsingular, we obtain

$$U = \frac{d}{ad - bc} M^{-1}.$$

Again from (4) and (6), we see that $-c(aMU + bMW) + a(cMU + dMW) = -cI_m$. Then, we have

$$W = -\frac{c}{ad - bc} M^{-1}.$$

Similarly, it follows from (5) and (7) that $-bI_m = d(aMV + bMZ) - b(cMV + dMZ) = (ad - bc)MV$ and $aI_m = -c(aMV + bMZ) + a(cMV + dMZ) = (ad - bc)MZ$. These result in

$$V = -\frac{b}{ad - bc} M^{-1} \quad \text{and} \quad Z = \frac{a}{ad - bc} M^{-1}.$$

Consequently, we have

$$N^{-1} = \frac{1}{ad - bc} \begin{bmatrix} dM^{-1} & -bM^{-1} \\ -cM^{-1} & aM^{-1} \end{bmatrix}.$$